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BI-QUOTIENT MAPS, COUNTABLY BI-SEQUENTIAL SPACES AND RELATED TOPICS

Roy C. OLSON *

Department of Mathematics, University of Washington, Seattle, Wash., U.S.A.

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Abstract: In a recent paper, E. Michael posed several problems regarding images of M-spaces and metrizable spaces under certain quotient maps. In this paper, some solutions are provided.

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bi-quotient	pointwise countable type	bi-quasi-k
hereditarily quotient	bi-k-space	relatively bi-quasi-k
bi-sequential	M-space	relative q-sequence
Fréchet space	q-space	equi-Lindelöf

1. Introduction

In [21], E. Michael investigated the images of certain spaces under the following types of maps [1], each generalizing the one preceding it: (1) open, (2) bi-quotient, (3) countably bi-quotient, (4) hereditarily quotient, and (5) quotient. Among these, bi-quotient maps, introduced by O. Hájek [13] and E. Michael [19], have particularly interesting properties, including the property that the cartesian product of bi-quotient maps is again bi-quotient [19]. In this paper, we provide answers to some of the questions raised in [21], one of the more interesting being the following result.

Theorem 1.1. *Let $f: X \to Y$ be a quotient map of a paracompact space X onto a compact Hausdorff space Y, with each $f^{-1}y$ Lindelöf. Then f is bi-quotient.*

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[1] Definitions will be provided in Section 2.

One of the main results of this paper, Theorem 4.4, is stated in a form that simultaneously generalizes Theorem 1.1 above and a theorem of V.V. Filippov [8, Theorem 2.1].

A special case of Theorem 5.1 is the following result, which was obtained independently by A.V. Arhangel'skiĭ and the author.

Theorem 1.2. *Every compact Hausdorff Fréchet space is countably bi-sequential.*

Corollary 1.3. *Every quotient map onto a compact Hausdorff Fréchet space is countably bi-quotient.*

In our examples, an important role is played by Example 6.5, due to J.R. Isbell, which is based on the assumption [2] that $2^{\aleph_0} < 2^{\aleph_1}$. Isbell has indicated that he has not planned to publish this example, so the proof is included here, with his kind permission. Other principal results include Examples 6.6 and 6.11, both of which are based on Example 6.5 and hence the assumption that $2^{\aleph_0} < 2^{\aleph_1}$. Example 6.6 shows that the product of two countably bi-sequential spaces need not be countably bi-sequential (or even a k-space). Example 6.11 shows that there exist a countably bi-sequential space X and a compact Hausdorff space Y such that $X \times Y$ (which must be a k-space) is not singly bi-k. Also, we observe that an example of Arhangel'skiĭ can be modified to obtain Example 6.1, a paracompact countably bi-sequential space which is not bi-sequential (or even bi-k), and whose existence depends only on the usual axioms of set theory.

In this paper, most of the definitions appear in Section 2, and some of the relationships between various spaces considered are made clear in Table 1. Columns D–F of this table are taken from Table I of [21], which the reader will find of additional interest and to which he is referred for further details.

The various sections after Section 2 of this paper are independent of each other, except that Section 4 uses results from Section 3, and Example 7.1 uses results from Section 6.

[2] Example 6.5 can also be constructed assuming Martin's Axiom [14] instead of $2^{\aleph_0} < 2^{\aleph_1}$. The continuum hypothesis is equivalent to Martin's Axiom + $(2^{\aleph_0} < 2^{\aleph_1})$.

2. Maps, filter bases, and spaces

All *maps* are continuous surjections. Topological spaces in general (including paracompact spaces) are not necessarily Hausdorff; but regular, completely regular, and normal spaces are Hausdorff in this paper. If $f: X \to Y$ is a map, and if $E \subset Y$, then f_E denotes $f | f^{-1}E$ (the restriction of f to $f^{-1}E$), considered as a map from $f^{-1}E$ to E.

Definition 2.1. Let $f: X \to Y$ be a map.

(a) f is *open* (*closed*) if the image of each open (closed) subset of X is an open (closed) subset of Y.

(b) f is *perfect* (*quasi-perfect*) if f is closed, and $f^{-1}y$ is compact (countably compact) for each $y \in Y$.

(c) f is *bi-quotient* (Michael [19]) (*countably bi-quotient* (Siwiec and Mancuso [25])) if for each $y \in Y$ every collection (countable collection) of open subsets of X which covers $f^{-1}y$ has a finite subcollection whose images cover some neighborhood of y.

(d) f is *hereditarily quotient* if for each subspace S of Y, f_S is a quotient map.

(e) f is *pseudo-open* (Arhangel'skiĭ [2], see also McDougle [15]) if for each $y \in Y$ and each neighborhood U of $f^{-1}y$, fU is a neighborhood of y.

(f) f is *quotient* if for $S \subset Y$, S is closed in Y whenever $f^{-1}S$ is closed in X.

Remark 2.2. Each type of map defined in Definition 2.1 is preserved under composition.

(a) Open maps and perfect maps are bi-quotient. Quasi-perfect maps are countably bi-quotient.

(b) If $f: X \to Y$ is countably bi-quotient, and if each $f^{-1}y$ is Lindelöf, then f is bi-quotient.

(c) The notions of hereditarily quotient and pseudo-open coincide [2, Theorem 1].

(d) A map $f: X \to Y$, Y Hausdorff, is bi-quotient if and only if $f \times i_Z$ is a quotient map for every space Z [13, Proposition 2; 19, Theorem 1.3], where i_Z denotes the identity map. This characterization of bi-quotient maps will not actually be used in this paper, but I think it is interesting and offers a justification for the name "bi-quotient".

It is sometimes useful to use characterizations of bi-quotient, countably bi-quotient and hereditarily quotient maps in terms of filter bases; see Remark 2.4(a)–(c).

Definition 2.3 (see [21, Section 1]). A *filter base* $\mathcal{F}$ is a non-empty collection of non-empty sets, such that if F_1 and F_2 belong to $\mathcal{F}$, then there exists some $F_3 \in \mathcal{F}$ with $F_3 \subset F_1 \cap F_2$.

In a topological space Y, a filter base $\mathcal{F}$ *accumulates* at a point y if $y \in \overline{F}$ for every $F \in \mathcal{F}$, and $\mathcal{F}$ *converges* to y if every neighborhood of y in Y contains some $F \in \mathcal{F}$.

Two filter bases $\mathcal{F}$ and $\mathcal{G}$ *mesh* if every $F \in \mathcal{F}$ intersects every $G \in \mathcal{G}$.

A filter base $\mathcal{F}$ *converges* to a set A in Y if every open subset of Y which contains A also contains some $F \in \mathcal{F}$.

A filter base $\mathcal{F}$ is an *outer network* at A in Y if $\mathcal{F}$ converges to A and $A \subset \bigcap \mathcal{F}$.

A decreasing sequence $\langle A_n \rangle$ of non-empty subsets of Y is a k-*sequence* (q-*sequence*) if it is an outer network at a compact (countably compact) subset of Y.

A decreasing sequence $\langle A_n \rangle$ of non-empty subsets of Y is a *relative* q-*sequence* [3] if $y_n \in A_n$ for all n implies that the sequence $\langle y_1, y_2, \ldots \rangle$ has an accumulation point [4] in Y.

Remark 2.4. (a) [19, Proposition 2.2] A map f: $X \to Y$ is bi-quotient if and only if, whenever a filter base $\mathcal{F}$ in Y accumulates at $y \in Y$, then $f^{-1}\ \mathcal{F}$ accumulates at some $x \in f^{-1}\ y$.

(b) [26, Proposition 3.2] A map f: $X \to Y$ is countably bi-quotient if and only if, whenever $\langle A_n \rangle$ is a decreasing sequence accumulating at y in Y, then $\langle f^{-1} A_n \rangle$ accumulates at some $x \in f^{-1} y$.

(c) [15, Lemma 1; 2, Theorem 1; 21, Lemma 5.2] A map f: $X \to Y$ is hereditarily quotient if and only if, whenever $y \in \overline{A}$ in Y, then $x \in (f^{-1}A)^{-}$ for some $x \in f^{-1} y$.

(d) Every k-sequence is a q-sequence, and every q-sequence is a relative q-sequence.

(e) If $\langle A_n \rangle$ is a relative q-sequence in Y, and $y_n \in A_n$ for all n, then every infinite subset of $\{y_1, y_2, \ldots\}$ must have an accumulation point in Y.

[3] The concept of "q-sequence" as used in this paper is that introduced by Nagata [24] and used by Michael [21]. The relative q-sequences of this paper are essentially the q-sequences of Morita and Rishel [23]. The term "relative q-sequence" is new, although the concept probably originated with the definition of q-space in [17].

[4] A point y is an *accumulation point* of the sequence $\langle y_n \rangle$ if every neighborhood of y contains y_n for infinitely many n.

Table 1

		D	E	F	G
1	domain of map / map	metrizable	paracon		
2	open	first-countable	pointwise countable type (Hausdorff)	strict q (regular)	q
3	bi-quotient	bi-sequential	bi-k	bi-quasi-k	relatively bi-quasi-k
4	countably bi-quotient	countably bi-sequential	countably bi-k	countably bi-quasi-k	relatively countably bi-quasi-k
5	hereditarily quotient	Fréchet	singly bi-k	singly bi-quasi-k	relatively singly bi-quasi-k
6	quotient	sequential	k (Hausdorff)	quasi-k (regular)	

We proceed to define the spaces considered in this paper, listed in Table 1. Except for column G, Table 1 is part of [21, Table I]. Most of the concepts in Table 1 will play a role in later sections; the others are included in the table for the sake of completeness. The reader might wish to postpone reading the definitions until the concepts appear later.

In Table 1, the entries in columns D, E, and F are actually characterized in [21] as images under the appropriate map (indicated at the beginning of the row in which the particular entry lies) of the spaces at the top of the respective columns, subject to the following restrictions, which are indicated parenthetically in Table 1: (i) A space of pointwise countable type is an open image of some paracompact M-space has only been established for Hausdorff spaces. (ii) A strict q-space is an open image of some M-space has only been established for regular spaces. (iii) Every quotient of a paracompact M-space is a k-space is only true for Hausdorff quotients. (iv) Every quotient of an M-space is a quasi-k-space is only true for regular quotients.

All entries in rows 2–6 are preserved by the corresponding maps, without assuming any separation properties. In column G, I do not know whether the entries in rows 3, 4 and 5 can be characterized as the appropriate images of the entries above them.

In each row, D → E → F → G, and in each column, 1 → 2 → 3 → 4 → 5 → 6, with the exceptions that the lowest entry in column E (column F) is implied by those above it only if the space is Hausdorff (regular). As Proposition 2.8 shows, columns F and G are the same in normal spaces (in rows 2–5) and columns E and F are identical in paracompact spaces.

We now define the spaces listed in Table 1. Our first definitions are those of the spaces in column D, except for metrizable spaces and first-countable spaces, which are well known.

Definition 2.5. (a) (E. Michael [21, Definition 3.D.1]) A space Y is *bi-sequential* if, whenever a filter base $\mathcal{F}$ accumulates at y in Y, then there is a decreasing sequence $\langle A_n \rangle$ in Y which meshes with $\mathcal{F}$ and converges to y.

(b) (F. Siwiec [26, Definition 1.1], E. Michael [21, Lemma 4.D.2]) A space Y is *countably bi-sequential* if, whenever $\langle F_n \rangle$ is a decreasing sequence of subsets of Y accumulating at y in Y, then there exist $y_n \in F_n$ such that the sequence $\langle y_n \rangle$ converges to y.

(c) A space Y is a *Fréchet space* if, whenever $y \in \overline{A}$ in Y, then there is a sequence in A which converges to y.

(d) (S.P. Franklin [10]) A space Y is *sequential* if a subset A of Y is closed whenever it has the following property: If $y_n \in A$ $(n = 1, 2, ...)$ and $y_n \to y$, then $y \in A$.

Our next definitions are those of the spaces in columns E, F, and G.

Definition 2.6. (a) (K. Morita [22, Theorem 6.1]) A space Y is a *paracompact* M-*space* (M-*space*) if there exist a metric space M and a perfect (quasi-perfect) map $f: Y \to M$. (See also E. Michael [21, Theorem 0.1].)

(b) A space Y is *of pointwise countable type* (A. Arhangel'skiĭ [3]) (a *strict* q-*space*; a q-*space* (E. Michael [21, Section 2.F])) if each point of Y has a k-sequence (q-sequence; relative q-sequence) of neighborhoods.

(c) (E. Michael [21, Section 3]) A space Y is *bi*-k (*bi-quasi*-k; *relatively bi-quasi*-k) if, whenever $\mathcal{F}$ is a filter base accumulating at y in Y, then there exists a k-sequence (q-sequence; relative q-sequence) $\langle A_n \rangle$ in Y which meshes with $\mathcal{F}$.

(d) (E. Michael [21, Section 4]) A space Y is *countably bi*-k (*countably bi-quasi*-k; *relatively countably bi-quasi*-k) if, whenever $\langle F_n \rangle$ is a decreasing sequence accumulating at y in Y, then there exists a k-sequence (q-sequence; relative q-sequence) $\langle A_n \rangle$ in Y such that $y \in (A_n \cap F_n)^-$ for all n.

(e) (E. Michael [21, Section 5]) A space Y is *singly bi*-k (*singly bi-quasi*-k; *relatively singly bi-quasi*-k) if, whenever $y \in \overline{F}$ in Y, then there exists a k-sequence (q-sequence; relative q-sequence) $\langle A_n \rangle$ in Y such that $y \in (A_n \cap F)^-$ for all n.

(f) A space Y is a k-*space* (*quasi*-k-*space* (J. Nagata [24])) if a subset A of Y is closed whenever $A \cap K$ is closed in K for every compact (countably compact) subset K of Y.

Lemma 2.7 (E. Michael). *Let Y be a normal space, and let $\langle A_n \rangle$ be a relative* q-*sequence in Y. Then there exists a* q-*sequence $\langle B_n \rangle$ such that $A_n \subset B_n$ for all n.*

Proof. First, let us show that $A = \bigcap_{n=1}^{\infty} A_n$ is countably compact. Suppose otherwise; then there exist distinct $y_n \in A$ $(n = 1, 2, \ldots)$ such that $D = \{y_1, y_2, \ldots\}$ has no accumulation points in A, a closed subset of Y. Therefore, D is a closed discrete subset of Y. Define $f: D \to \mathbf{R}$ (reals) by $fy_n = n$. By the Tietze Extension Theorem, f has a continuous extension $\tilde{f}: Y \to \mathbf{R}$. Let $U_n = \tilde{f}^{-1}(n-1, n+1)$; then U_n is a neighborhood of y_n, hence $U_n \cap A_n \neq \emptyset$. Let $x_n \in U_n \cap A_n$. Because $\langle A_n \rangle$ is a relative q-sequence, the sequence $\langle x_n \rangle$ has an accumulation point $x \in Y$. Hence $\tilde{f}x$ is an accumulation point of the sequence $\langle \tilde{f}x_n \rangle$. But this is impossible since $\langle \tilde{f}x_n \rangle$ clearly has no accumulation points in $\mathbf{R}$. Therefore, A is countably compact.

It is now clear that if $B_n = A_n \cup A$, then $\langle B_n \rangle$ is an outer network at A, and hence a q-sequence.

Proposition 2.8. *Let Y be a topological space.*

(i) *If Y is paracompact, then in each of rows 1–6 of Table 1,* E $\leftrightarrow$ F.

(ii) *If Y is normal, then in each of rows 3–5 of Table 1,* F $\leftrightarrow$ G.

(ii′) *If Y is regular, then* F $\leftrightarrow$ G *in row 2.*

(iii) *If Y is paracompact and Hausdorff, then in each of rows 2–5 of Table 1,* E $\leftrightarrow$ *G.*

Proof. (i) Because any locally finite open cover of a countably compact set is finite and a closed subset of a paracompact space is paracompact, the closure of any countably compact subset of a paracompact space is compact. Hence if $\langle A_n \rangle$ is a q-sequence in a paracompact space Y, let $B_n = A_n \cup (\bigcap_{k=1}^{\infty} A_k)^-$. Then $\langle B_n \rangle$ is a k-sequence in Y such that $A_n \subset B_n$ for all n. It follows that F $\rightarrow$ E in each of rows 1–6. (This result was proved in [21, Section 0] for paracompact Hausdorff spaces.)

(ii) This follows from Lemma 2.7.

(ii′) E. Michael has made this observation in [21, Section 2.F].

(iii) This follows from (i), (ii), and (ii′).

In view of Proposition 2.8 (ii′), it seems plausible that regularity, instead of normality, might be sufficient for Lemma 2.7 or Proposition 2.8 (ii); see Problem 2.9. However, Hausdorff does not suffice for Proposition 2.8 (ii), (ii′) or (iii), hence neither for Lemma 2.7; see Example 7.6. For the question E $\leftrightarrow$ F, see [21].

Problem 2.9. In Proposition 2.8 (ii), can "Y is normal" be replaced by "Y is regular"? (Lemma 2.7 would not be true if "Y is normal" were replaced by "Y is regular": In the space Ψ, described in Example 7.6, let $A_n = \mathrm{N}$ ($n = 1, 2, ...$). Then $\langle A_n \rangle$ is a relative q-sequence, but there is no q-sequence $\langle B_n \rangle$ in Ψ such that $A_n \subset B_n$ for all n.)

Another condition that yields some equivalences in Table 1 and which we need later on, is provided by the following result, most of which is from E. Michael's [21, Theorem 7.3a].

Proposition 2.10. *Let Y be a Hausdorff space in which every point is a G_δ. Then in each of rows 2–6 of Table 1,* D $\leftrightarrow$ E; *if Y is regular, then in each of rows 2–6, all columns are equivalent.*

Proof. Except for the part dealing with column G, this is Michael's [21, Theorem 7.3a]. Let us show that in each of rows 2–5 of Table 1, G $\rightarrow$ E. For, if $y \in Y$ is an accumulation point of a relative q-sequence $\langle A_n \rangle$, and if $\langle B_n \rangle$ is a decreasing sequence of closed neighborhoods of y such that $\bigcap_{n=1}^{\infty} B_n = \{y\}$, then $\langle A_n \cap B_n \rangle$ is (an outer network at $\{y\}$ and hence) a k-sequence in Y which meshes with any filter base $\mathcal{F}$ for which $y \in (A_n \cap F)^-$ for all $F \in \mathcal{F}$.

3. Equi-Lindelöf collections

Definition 3.1. A collection $\mathscr{A}$ of subsets of a topological space X is *equi-Lindelöf* if every open cover $\mathcal{U}$ of X has an open refinement $\mathcal{V}$ such that each $A \in \mathscr{A}$ intersects only countably many $V \in \mathcal{V}$.

The notion of an *equi-Lindelöf* collection is a new concept, to be used in Section 4, but is also of interest in itself. The term "equi-Lindelöf" was suggested by E. Michael.

Remark 3.2. (a) If $E \subset X$ and $\overline{E}$ is Lindelöf, then any collection of subsets of E is equi-Lindelöf in X.

(b) If $\mathscr{A}$ is an equi-Lindelöf collection of subsets of X and $A \in \mathscr{A}$, then $\overline{A}$ is Lindelöf.

(c) If $\mathscr{A}$ is an equi-Lindelöf collection of subsets of X, and if $\mathscr{B}$ is a collection of subsets of X such that each $B \in \mathscr{B}$ is a subset of $\overline{A}$ for some $A \in \mathscr{A}$, then $\mathscr{B}$ is equi-Lindelöf.

(d) If $\mathscr{A}$ is an equi-Lindelöf collection of subsets of X, and if $\bigcup \mathscr{A}$ is closed in X, then $\bigcup \mathscr{A}$ is meta-Lindelöf [5].

(e) If $\mathscr{A}$ is a countable equi-Lindelöf collection of subsets of X, then $\{\bigcup \mathscr{A}\}$ is equi-Lindelöf, and by Remark 3.2(b), $(\bigcup \mathscr{A})^-$ is Lindelof.

Proposition 3.3. *If X is paracompact, then any collection of Lindelöf subsets of X is equi-Lindelöf.*

Proof. Let $\mathscr{A}$ be a collection of Lindelöf subsets of X, and let $\mathcal{U}$ be an open cover of X. Since X is paracompact, $\mathcal{U}$ has a locally finite open refinement $\mathcal{V}$. However, any locally finite open cover of a Lindelöf space is countable, so each $A \in \mathscr{A}$ intersects only countably many $V \in \mathcal{V}$.

Proposition 3.4. *If X is meta-Lindelöf, then any collection of separable subsets of X is equi-Lindelöf.*

Proof. This follows easily, as in Proposition 3.3, from the fact that a point-countable open cover of a separable space is countable.

In contrast to Propositions 3.3 and 3.4, it is not true that in a meta-Lindelöf space any collection of Lindelöf subsets is equi-Lindelöf, as the following example demonstrates.

[5] A space is *meta-Lindelöf* if every open cover has a point-countable open refinement.

Example 3.5 (E. Michael). There exists a topological space Z which has a point-countable base (hence is meta-Lindelöf) and a countable collection of Lindelöf subsets which is not equi-Lindelöf.

Proof. E. Michael [16] shows that spaces X and Y exist which satisfy the following conditions.

(i) X is a regular Lindelöf space with point-countable base.

(ii) Y is a separable metric space.

(iii) $X \times Y$ has a point-countable base, but is not normal.

Let $\{y_1, y_2, \ldots\}$ be dense in Y. Then $\mathscr{A} = \{X \times \{y_n\}\}_{n=1}^{\infty}$ is a countable collection of Lindelöf subsets of the space $Z = X \times Y$, which has a point-countable base. If $\mathscr{A}$ were equi-Lindelöf in Z, then $(\bigcup \mathscr{A})^{-} = Z$ would be Lindelöf, by Remark 3.2(e). But Z is not Lindelöf since Z is regular but not normal. That completes the proof.

4. Quotient maps onto compact (and some more general) spaces

In this section, we establish some sufficient conditions for a quotient map to be bi-quotient or hereditarily quotient, providing examples to show certain conditions are not sufficient, and we consider other conditions in some problems.

First, let us introduce some preliminary results, which may be of independent interest.

Lemma 4.1. *Let $\mathscr{F}$ be a point-countable cover of a set Y. If Y cannot be covered by finitely many $F \in \mathscr{F}$, then Y has a countable subset which cannot be covered by finitely many $F \in \mathscr{F}$.*

Proof. For each $y \in Y$, Let $F_1(y), F_2(y), \ldots$ denote the members of $\mathscr{F}$ that contain y. Consider the subset $C = \{y_1, y_2, \ldots\}$ of Y defined as follows: Let $y_1 \in Y$ be arbitrary. If $n \geqslant 2$ and $y_1, \ldots, y_{n-1}$ have been chosen, let $y_n \in Y \setminus \bigcup_{i,j<n} F_i(y_j)$. It is easily checked that C has the required properties.

Lemma 4.2. *Let $f: X \to Y$ be a quotient map, and $\mathscr{A}$ a collection of closed subsets of Y. If $f^{-1}\mathscr{A}$ is locally finite, then $\mathscr{A}$ is locally finite.*

Proof. It is easy to check that $\mathscr{A}$ is point-finite. If $y \in Y$, Let $\mathscr{A}' = \{A \in \mathscr{A} : y \notin A\}$, and note that $\mathscr{A} \setminus \mathscr{A}'$ is finite. Then $f^{-1}\mathscr{A}'$ is a locally finite collection of closed subsets of X, hence $\bigcup f^{-1}\mathscr{A}' = f^{-1}(\bigcup \mathscr{A}')$ is closed in X. Since f is quotient, $\bigcup \mathscr{A}'$ is closed in Y. Thus $Y \setminus \bigcup\mathscr{A}'$ is a neighborhood of y which intersects only finitely many $A \in \mathscr{A}$. That completes the proof.

Theorem 4.3. *Let $f: X \to Y$ be a quotient map onto a Hausdorff space Y, and let $E \subset Y$ with $\{f^{-1}z: z \in E\}$ equi-Lindelöf in X. If $\langle A_n \rangle$ is a relative* q*-sequence in Y of subsets of E which accumulates at $y \in E$, then $\langle f^{-1}A_n \rangle$ accumulates at some $x \in f^{-1}y$.*

Proof. Suppose otherwise. For $n = 1, 2, \ldots$, let $U_n = X \setminus (f^{-1}A_n)^-$. Then $\{U_n\}_{n=1}^\infty$ is an increasing open cover of $f^{-1}y$ with each fU_n intersecting only finitely many A_k.

For each $z \in Y \setminus \{y\}$, let V_z be an open neighborhood of z whose closure does not contain y, and let $\mathcal{V}$ be an open refinement of the open cover $\{U_n\}_{n=1}^\infty \cup \{f^{-1} V_z: z \in Y \setminus \{y\}\}$ of X such that, for each $z \in E$, $f^{-1}z$ intersects only countably many $V \in \mathcal{V}$.

Next, observe that A_n cannot be covered by finitely many fV with $V \in \mathcal{V}$. In fact, if $\mathcal{F}$ is a finite subcollection of $\mathcal{V}$, then it is easy to verify that y belongs to the closure of $A_n \setminus \bigcup f\mathcal{F}$.

Now, $f\mathcal{V}$ is a point-countable cover of A_n. By Lemma 4.1, there exists a countable subset B_n of A_n which cannot be covered by finitely many members of $f\mathcal{V}$. Because $\bigcup_{n=1}^\infty B_n$ is a countable subset of E, $f^{-1}(\bigcup_{n=1}^\infty B_n)$ intersects only countably many $V \in \mathcal{V}$, say, $V_1, V_2, \ldots$. Choose $y_n \in B_n \setminus (fV_1 \cup \ldots \cup fV_n)$. Observe that since $\bigcup_{n=1}^\infty B_n \subset \bigcup_{n=1}^\infty fV_n$, there are infinitely many y_n. The collection $\{\{y_1\}, \{y_2\}, \ldots\}$ of closed subsets of Y has the property that for each $V \in \mathcal{V}$, fV intersects only finitely many members of the collection; so by Lemma 4.2, the collection is locally finite, and hence $\{y_1, y_2, \ldots\}$ has no accumulation points in Y. But this contradicts Remark 2.4(e), since $y_n \in A_n$ for all n, and $\langle A_n \rangle$ is a relative q-sequence. That completes the proof.

The following result has particular interest in the special case where $E = Y$ and Y is compact.

Theorem 4.4. *Let $f: X \to Y$ be a quotient map onto a Hausdorff space Y which is relatively countably bi-quasi-*k*, and let E be a subset of Y such that $\{f^{-1}y: y \in E\}$ is equi-Lindelöf in X. Then f_E is bi-quotient.*

Proof. In view of Remarks 3.2(b) and 2.2(b), it is enough to prove that

f_E is countably bi-quotient, and we use the characterization in Remark 2.4(b).

Suppose $\langle F_n \rangle$ is a decreasing sequence of subsets of E which accumulates at $y \in E$. Then there is a relative q-sequence $\langle A_n \rangle$ in Y with $y \in (F_n \cap A_n)^-$ for all n. Since $\langle A_n \rangle$ is a relative q-sequence, so is $\langle F_n \cap A_n \rangle$. By Theorem 4.3, $\langle f^{-1}(F_n \cap A_n) \rangle$ accumulates at some $x \in f^{-1}y$. Therefore $\langle f^{-1}F_n \rangle$ accumulates at $x \in f^{-1}y$. That completes the proof.

The following is proved like Theorem 4.4, using the characterization in Remark 2.4(c) instead of Remark 2.4(b).

Theorem 4.5. *Let $f: X \to Y$ be a quotient map onto a relatively singly bi-quasi-*k *Hausdorff space Y, and let E be a subset of Y such that $\{f^{-1}y: y \in E\}$ is equi-Lindelöf in X. Then f_E is hereditarily quotient.*

The remainder of this section is devoted to applications and possible modifications of Theorem 4.4. Observe that, for the space Y, we assume the weakest condition in row 4 of Table 1, but even the strongest condition (Fréchet) in row 5 is not sufficient, as [19, Example 8.1] shows.

Before continuing, we pause to raise a question. First, we need the following lemma.

Lemma 4.6. *Let Y be a* T_1*-space. Then* (i) *implies* (ii) *below.*

(i) *Y is a relatively countably bi-quasi-*k*-space.*

(ii) *Whenever $\langle F_n \rangle$ is a decreasing sequence with common accumulation point y in Y (that is, $y \in (F_n \setminus \{y\})^-$ for all n), then there exist $A_n \subset F_n$ which are closed in Y such that $\bigcup_{n=1}^{\infty} A_n$ is not closed in Y.*

Proof. That every countably bi-quasi-k T_1-space satisfies (ii) is the content of Michael's [21, Lemma 9.1], and an obvious modification of Michael's proof shows that (i) implies (ii).

In general, (ii) does not imply (i); see [21, Example 10.10] or Example 7.1. Nevertheless, we have the following problem.

Problem 4.7. In Theorem 4.4, can the condition that Y satisfy Lemma 4.6(i) be replaced by the condition that Y satisfy 4.6(ii)? What if $E = Y$? (E. Michael [21, Theorem 9.5] has shown that the answer is affirmative if X or Y is determined by countable subsets (a property possessed, in particular, by all sequential spaces). Note that by Remarks 3.2(a), (e), Michael's condition that $(f^{-1}E)^{-}$ is Lindelöf for countable E is equivalent to the condition that $\{f^{-1}y: y \in E\}$ is equi-Lindelöf for such E.)

One consequence of Theorem 4.4 is Theorem 1.1 of the Introduction, in view of Proposition 3.3. Another consequence is the following result, due to V.V. Filippov [8, Theorem 2.1].

Theorem 4.8 (V.V. Filippov). *Let $f: X \to Y$ be a quotient s-mapping* [6] *of a space X with point-countable base onto a Hausdorff space Y of pointwise countable type. Then the map f is bi-quotient.*

Proof. A space with point-countable base is clearly meta-Lindelöf, and since f is an s-mapping, $\{f^{-1}y: y \in Y\}$ is equi-Lindelöf in X, by Proposition 3.4. Finally, since spaces of pointwise countable type are relatively countably bi-quasi-k, Theorem 4.4 applies; therefore, f is bi-quotient.

Problem 4.9. Does Filippov's theorem, Theorem 4.8, remain true if the condition "f is an s-mapping" is replaced by "$f^{-1}y$ is Lindelöf for each $y \in Y$"? (See Example 3.5.)

In a different direction, Michael [21, Problem 5] asked, in effect, whether in Theorem 4.8, the condition that X has a point-countable base can be weakened to the condition that X is first-countable. Example 4.10 shows that the answer is no, even if Y is compact.

Example 4.10. There exists a quotient map $f: X \to Y$, with X locally compact Hausdorff, first-countable, and each $f^{-1}y$ compact, with Y compact Hausdorff, but f not hereditarily quotient.

Proof. E. Michael [21, Example 10.13] has shown that there exists a quotient map $h: X^* \to Y$ with X^* lcoally compact Hausdorff and bi-sequential and each $h^{-1}y$ finite, with Y compact Hausdorff, but h not hereditarily quotient. Indeed, X^* is a disjoint union $X^* = X_1 + X_2$, where X_1 is a first-countable, locally compact Hausdorff space [7] and X_2 is the one-point compactification of a discrete space D of cardinality $2^{\aleph_0}$, and $h|X_1$ and $h|X_2$ are both one-to-one. By Alexandroff and Ury-

[6] An *s-mapping* $f: X \to Y$ is a map for which $f^{-1}y$ is separable for every $y \in Y$.

[7] In fact, the space X_1 is the space Ψ of Isbell [12, 51, p. 79] (see Example 7.6), and the space Y is the one-point compactification of Ψ, studied by Franklin [11, Example 7.1].

sohn [1] (see [7, Section 1]), D can be embedded as a dense open subset of a first-countable compact Hausdorff space Z (called $A(I)$ in [7]), with $Z \setminus D$ homeomorphic to the interval $I = [0, 1]$.

Let $g_1 : X_1 \to X_1$ be the identity map, and let $g_2 : Z \to X_2$ be the continuous extension of the identity map on D. Let $g : X_1 + Z \to X_1 + X_2$ be defined by $g \mid X_1 = g_1$, $g \mid Z = g_2$.

Let $X = X_1 + Z$, and define $f: X \to Y$ by $f = h \circ g$. Then f has the required properties; in particular, f is not hereditarily quotient, for if it were, then h would also be, by the following lemma.

Lemma 4.11. *Let $f: X \to Y$ and $g : Y \to Z$ be maps. If $g \circ f$ is open, bi-quotient, countably bi-quotient, hereditarily quotient, or quotient, then so is g.*

Proof. Each of these claims is routinely verified. We remark that this is well known for quotient maps, and explicitly stated [19, Proposition 3.6] [8] for bi-quotient maps.

Remark 4.12. The example of Michael [21, Example 10.13], which was used in the proof of our Example 4.10, has all the features of Example 4.10 except first-countability of X, and moreover the map in that example is finite-to-one (in fact, two-to-one). Repeating a problem of Michael [21, Problem 5], we ask:

Problem 4.13. Can Example 4.10 be strengthened to make f finite-to-one (or even two-to-one) ?

5. Every compact Hausdorff Fréchet space is countably bi-sequential

In [21, Problem 3], E. Michael asks whether a compact Hausdorff Fréchet space must be countably bi-sequential. Independently, A.V. Arhangel'skiĭ and the author have found this is indeed the case. In fact, the condition that the space be compact Hausdorff may be weakened considerably, as the following theorem demonstrates.

Theorem 5.1. *Every regular Fréchet space Y which satisfies Lemma 4.6 (ii) is countably bi-sequential.*

[8] In this reference, $f \circ g$ should be $g \circ f$, a misprint.

Proof. Suppose, $\langle F_n \rangle$ is a decreasing sequence of subsets of Y, and $y^* \in \bar{F}_n$ for all $n = 1, 2, \ldots$. Define

$$S = \{y \in Y : \text{there exists } y_n \in F_n \text{ with } y_n \to y\} .$$

We shall show that $y^* \in S$. First, let us show S is closed.

If $y' \in \bar{S}$, then, because Y is Fréchet, there is a sequence $\langle y'_n \rangle$ in S which converges to y'. From the definition of S, there exist $y^k_n \in F_k$ such that $y^k_n \to y'_n$, for all n. Consequently, y' belongs to the closure of $\{y^k_n\}_{k \geqslant n \geqslant 1}$. Because Y is Fréchet, there is a sequence $\langle y^{k(m)}_{n(m)} \rangle$, with $k(m) \geqslant n(m)$ for $m = 1, 2, \ldots$, which converges to y'. If some $y'_n = y'$, then $y' \in S$; otherwise we may assume $n(m) \geqslant m$. But then $k(m) \geqslant m$, so $y^{k(m)}_{n(m)} \in F_{k(m)} \subset F_m$, and hence $y' \in S$. Thus S is closed.

If $y^* \notin S$, then, since Y is regular, there exists a neighborhood U of y^* satisfying $\bar{U} \cap S = \emptyset$. Then $y^* \in (U \cap F_n)^-$ for all n, but Y satisfies Lemma 4.6(ii), so there exist closed subsets $A_n \subset U \cap F_n$ such that $\bigcup_{n=1}^{\infty} A_n$ is not closed. Let $y \in (\bigcup_{n=1}^{\infty} A_n)^- \setminus \bigcup_{n=1}^{\infty} A_n$. Then there exist $y_n \in \bigcup_{k=1}^{\infty} A_k$ such that $y_n \to y$; since $y \notin A_k$ and A_k is closed, we can even choose $y_n \in \bigcup_{k=n}^{\infty} A_k$. Hence $y_n \in U \cap F_n$ and $y_n \to y$. This implies $y \in S \cap \bar{U} = \emptyset$, and the contradiction completes the proof.

Corollary 5.2. *If $f: X \to Y$ is quotient, and if Y is a regular Fréchet space which satisfies Lemma 4.6(ii), then f is countably bi-quotient.*

Proof. This follows immediately from Theorem 5.1 and a result of Siwiec [26, Theorem 4.3] (see also [21, Proposition 8.1b]) which states that a quotient map onto a Hausdorff countably bi-sequential space is countably bi-quotient.

6. Countably bi-sequential spaces and their products

Let us first discuss some examples of countably bi-sequential spaces that are not bi-sequential.

Assuming the existence of a measurable cardinal, E. Michael [21, Example 10.15] has an example of a compact Hausdorff space which is countably bi-sequential but not bi-sequential.

Assuming $2^{\aleph_0} < 2^{\aleph_1}$, J.R. Isbell has an example (Example 6.5 below) of a countably bi-sequential space which is countable and regular (hence paracompact) but not bi-sequential.

A.V. Arhangel'skiĭ has an example of a sequentially compact space which is countably bi-sequential but not bi-sequential, whose existence depends only on the usual axioms of set theory (including the axiom of choice). We now show that this example can be modified to obtain such a space that is paracompact (but no longer sequentially compact).

Example 6.1. There exists a paracompact countably bi-sequential space Y, with only one non-isolated point, that is not bi-sequential and whose existence depends only on the usual axioms of set theory.

Proof. Let X be the space of Arhangel'skiĭ that is countably bi-sequential but not bi-sequential, and let $x \in X$ be a point for which there exists a filter base $\mathcal{F}$ which accumulates at x but no decreasing sequence in X both meshes with $\mathcal{F}$ and converges to x. Let Y be the space obtained from X by defining a subset U of Y to be *open* in Y if either (i) U is open in X, or (ii) $x \notin U$. Then it is easy to check that Y has the required properties.

Problem 6.2 (see [21, Problem 4]). Is there a regular Lindelöf, or even compact Hausdorff space with all the features of Example 6.1? Is there a separable example with these properties?

Remark 6.3. At this point, we add a word or two concerning the above countably bi-sequential spaces which are not bi-sequential, and consider the question of whether they are bi-k, bi-quasi-k, or relatively bi-quasi-k.

(a) Michael's example is compact Hausdorff, hence bi-k.

(b) Becaus[illegible] example is countable and regular each point is a G_δ; therefore [illegible]ple is not a bi-quasi-k-space, by Proposition 2.10. In fact, it is not [illegible] relatively bi-quasi-k.

(c) Arhangel'skiĭ has shown that his example is not bi-k; however, because his example is sequentially compact, it is clearly an M-space and hence bi-quasi-k.

(d) If Y is the space of Example 6.1, and if the non-isolated point of Y is taken to be a point of Arhangel'skiĭ's example X at which X is not bi-k, then Y is not bi-k either, since any compact subset of Y must be compact in X. Because Y is paracompact and Hausdorff, Y is not even relatively countably bi-quasi-k, by Proposition 2.8.

Regarding the remaining examples in this section, subspaces of βN are of particular interest, so we consider some preliminaries concerning βN.

Remark 6.4. The space βN is the Stone–Čech compactification of the set N of positive integers with the discrete topology.

(a) If $A \subset \beta$N, then A^β denotes the closure of A in βN.

(b) If A is an infinite subset of N, then A' denotes $A^\beta \setminus$ N.

(c) $B' \subset A'$ if and only if $B \setminus A$ is finite.

(d) Every open-closed set in βN is of the form A^β, for some $A \subset$ N. The sets A^β form a base for the open sets in βN.

(e) Every open-closed set in βN $\setminus$ N is of the form A'. The sets A' form a base for the open sets in βN $\setminus$ N.

(f) Every non-empty G_δ in βN $\setminus$ N has a non-empty interior.

(g) In βN $\setminus$ N, two disjoint open F_σ subsets have disjoint closures.

(h) In βN and in βN $\setminus$ N, a zero set (of a continuous function into the reals) is the same as a closed G_δ; and such a set can be written as the countable intersection of a decreasing sequence of open-closed sets.

Of the above, (b)–(g) are from Gillman and Jerison [12, 6S, pp. 98–99 and 14N(4), p. 215]. The notation for (a) was suggested by John Isbell. Finally, (h) holds for any compact Hausdorff space in which the open-closed sets form a base (normality is sufficient for the first part of (h)).

The following example of J.R. Isbell, which depends on the assumption that $2^{\aleph_0} < 2^{\aleph_1}$, is fundamental to our constructions. The examples which depend on this assumption are indicated by ($2^{\aleph_0} < 2^{\aleph_1}$). Since Isbell has indicated that he does not plan to publish the following example, the proof is included here, with his kind permission.

Example 6.5 (J.R. Isbell ($2^{\aleph_0} < 2^{\aleph_1}$)). In βN $\setminus$ N there are closed subsets A and B with non-empty intersection, such that neither intersects the interior of the other, with the following properties.

(i) Every zero set which intersects A intersects the interior of A.

(ii) Every zero set which intersects B intersects the interior of B.

(iii) The (regular) quotient spaces $Y = (\mathrm{N} \cup A)/A$ and $Z = (\mathrm{N} \cup B)/B$ are countably bi-sequential but not bi-sequential.

Proof. *Part* I. Let us first show that (iii) follows from (i) and (ii). Write $Y = \mathrm{N} \cup \{A\}$ and $Z = \mathrm{N} \cup \{B\}$. Because points of N are open in Y, every subset of Y which contains A is closed, and Y is a regular space. Likewise, Z is regular.

We now show that Y is countably bi-sequential. Let $\langle F_n \rangle$ be a decreasing sequence in Y which accumulates at some $p \in Y$. We must show that there is a sequence $\langle y_n \rangle$, with $y_n \in F_n$ for all n, such that $y_n \to p$. If

$p \in F_n$ for all n, then there is nothing to prove. Thus we assume $p = A$ and $F_n \subset \mathbf{N}$, for all n. Since $A \in \overline{F}_n$, the closure of F_n in Y, $A \cap F'_n \neq \emptyset$ in $\beta\mathbf{N} \setminus \mathbf{N}$. Hence $C = \bigcap_{n=1}^{\infty} F'_n$ is a closed G_δ, and therefore a zero set, in $\beta\mathbf{N} \setminus \mathbf{N}$ which intersects A. By hypothesis, C intersects the interior of A, so this intersection is a non-empty G_δ in $\beta\mathbf{N} \setminus \mathbf{N}$ with non-empty interior (by Remark 6.4(f)). Therefore, there exists $F \subset \mathbf{N}$ such that $\emptyset \neq F' \subset A \cap C$. Since $F' \subset F'_n$, $F \cap F_n$ must be infinite. Choose distinct $y_n \in F \cap F_n$. Then $y_n \to A$ in Y. Similarly, Z is countably bi-sequential.

That neither Y nor Z is bi-sequential follows from Example 6.6 and Proposition 6.7 (and the above paragraph). Nevertheless, we give a direct proof. To show that Y is not bi-sequential, let $x \in A \cap B$, and consider the filter base $\mathcal{F} = \{F \subset \mathbf{N}: x \in F'\}$. Clearly $\mathcal{F}$ accumulates at A in Y, hence if Y is bi-sequential, then there exists a decreasing sequence $\langle A_n \rangle$ of subsets of Y which meshes with $\mathcal{F}$ and converges to A in Y. Let $F_n = A_n \cap \mathbf{N}$. Then $\langle F_n \rangle$ also meshes with $\mathcal{F}$ and converges to A in Y. Now $\mathcal{F}$ is a maximal filter base in $\mathbf{N}$, hence $F_n \in \mathcal{F}$ for all n. Because $\langle F_n \rangle$ converges to A in Y, it follows that every open subset of $\beta\mathbf{N}$ which contains A contains some F_n^β. Hence $\bigcap_{n=1}^{\infty} F'_n \subset A$, and $\bigcap_{n=1}^{\infty} F'_n$ is a zero set containing x. By (ii), $\bigcap_{n=1}^{\infty} F'_n$ intersects the interior in $\beta\mathbf{N} \setminus \mathbf{N}$ of B; but this implies A intersects the interior of B, contrary to assumption. Therefore Y is not bi-sequential. Likewise Z is not bi-sequential.

Part II. Let us now construct two closed subsets A and B of $\beta\mathbf{N} \setminus \mathbf{N}$, neither intersecting the interior of the other, A and B having non-empty intersection and satisfying (i) and (ii). Let ω_1 denote the first uncountable ordinal, and suppose $2^{\aleph_0} < 2^{\aleph_1}$.

Since non-empty G_δ subsets of $\beta\mathbf{N} \setminus \mathbf{N}$ have non-empty interior (Remark 6.4 (f)), there exists a collection $\{R_\alpha\}_{\alpha<\omega_1}$ of non-empty open-closed subsets of $\beta\mathbf{N} \setminus \mathbf{N}$ such that whenever $\alpha < \beta < \omega_1$, $R_\alpha \supset R_\beta$ and $R_\alpha \setminus R_\beta \neq \emptyset$. The differences $D_\alpha = R_\alpha \setminus R_{\alpha+1}$ are $\aleph_1$ disjoint open-closed subsets of $\beta\mathbf{N} \setminus \mathbf{N}$. There are $2^{\aleph_1}$ subsets J of the index set $\{\alpha: \alpha < \omega_1\}$, and $2^{\aleph_0}$ continuous functions from $\beta\mathbf{N} \setminus \mathbf{N}$ into the reals ($\beta\mathbf{N} \setminus \mathbf{N}$ is a closed subspace of the separable normal space $\beta\mathbf{N}$); so for some J, there is no continuous function from $\beta\mathbf{N} \setminus \mathbf{N}$ into the reals which separates $U = \bigcup\{D_\alpha: \alpha \in J\}$ from $V = \bigcup\{D_\alpha: \alpha \notin J\}$. Hence $\overline{U}$, the closure of U in $\beta\mathbf{N} \setminus \mathbf{N}$, intersects $\overline{V}$. For $\mu < \omega_1$, let $U_\mu = U \setminus R_\mu$ and $V_\mu = V \setminus R_\mu$. That is,

$$U_\mu = \bigcup\{D_\alpha: \alpha \in J, \alpha < \mu\}, \qquad V_\mu = \bigcup\{D_\alpha: \alpha \notin J, \alpha < \mu\}.$$

Using transfinite induction, we will construct increasing families $\langle U'_\mu \rangle_{\mu<\omega_1}$

and $\langle V'_\mu \rangle_{\mu<\omega_1}$ of open-closed subsets of $\beta\mathrm{N}\setminus\mathrm{N}$ such that for all $\mu < \omega_1$

$$U'_\mu \supset U_\mu, \quad V'_\mu \supset V_\mu, \quad U'_\mu \cap V'_\mu = \emptyset.$$

To this end, we will inductively construct open-closed U'_μ containing U_μ and disjoint from $V \cup R_\mu$, and define $V'_\mu = (\beta\mathrm{N} \setminus \mathrm{N}) \setminus (U'_\mu \cup R_\mu)$. Note V'_μ contains V_μ. Let $U'_0 = U_0 = \emptyset$; at an isolated ordinal $\lambda + 1$ let $U'_{\lambda+1} = U'_\lambda \cup U_{\lambda+1}$. At a limit ordinal λ, $\bigcup\{U'_\kappa : \kappa < \lambda\}$ and $R_\lambda \cup \bigcup\{V'_\kappa : \kappa < \lambda\}$ are disjoint (by induction) open F_σ subsets of $\beta\mathrm{N}\setminus\mathrm{N}$; so they have disjoint closures (by Remark 6.4(g)), and hence there is an open-closed set U'_λ containing $\bigcup\{U'_\kappa : \kappa < \lambda\} \supset \bigcup\{U_\kappa : \kappa < \lambda\} = U_\lambda$ and disjoint from $R_\lambda \cup \bigcup\{V'_\kappa : \kappa < \lambda\} \supset R_\lambda \cup V_\lambda = R_\lambda \cup V$. Observe that if $\alpha < \beta < \omega_1$, then $U'_\alpha \subset U'_\beta$ and $V'_\alpha \subset V'_\beta$. Thus $\langle U'_\mu \rangle_{\mu<\omega_1}$ and $\langle V'_\mu \rangle_{\mu<\omega_1}$ are as required.

Define $A \supset U$ and $B \supset V$ as follows:

$$A = (\bigcup\{U'_\mu : \mu < \omega_1\})^-, \qquad B = (\bigcup\{V'_\mu : \mu < \omega_1\})^-.$$

Thus A and B are closed, and $A \cap B \supset \overline{U} \cap \overline{V} \neq \emptyset$. Since A and B are closures of two disjoint open sets, neither intersects the interior of the other (if the interior of B intersects A, then it intersects $\bigcup_\mu U'_\mu$ which must, in turn, intersect $\bigcup_\mu V'_\mu$).

Finally, we verify (i) and (ii). Suppose $\bigcap_{n=1}^\infty F_n$ is a closed G_δ (with each F_n open-closed in $\beta\mathrm{N}\setminus\mathrm{N}$; see Remark 6.4(h)) which intersects A. For each n, there exists $\mu_n < \omega_1$ such that $F_n \cap U'_{\mu_n} \neq \emptyset$. Let $\mu = \sup \mu_n$; then $\bigcap_{n=1}^\infty F_n \cap U'_\mu$ is a non-empty subset of U'_μ, and thus lies in the interior of A. Hence (i) is satisfied. Similarly, (ii) is satisfied. That completes the proof.

John Isbell established Example 6.5 in a letter to Ernest Michael. Isbell also observed that if one assumes the continuum hypothesis, which is slightly stronger than $2^{\aleph_0} < 2^{\aleph_1}$, then Example 6.5 (i) and (ii) can easily be obtained from a result of Fine and Gillman [9, Theorem 4.6d]. As the above proof indicates, Example 6.5 (iii) follows from Example 6.5 (i) and (ii).

The remainder of this section is concerned with products of spaces in which at least one of the factors is countably bi-sequential. E. Michael proved [21, Proposition 3.D.3] that the product of two (or even countably many) bi-sequential spaces is always bi-sequential, and asked [21, Problem 2] whether an analogous result is true for countably bi-sequential spaces. Using Example 6.5, we shall now show that, if $2^{\aleph_0} < 2^{\aleph_1}$, the answer is negative.

Example 6.6 ($2^{\aleph_0} < 2^{\aleph_1}$). There exist countable, regular, countably bi-sequential spaces Y and Z such that $Y \times Z$ is not countably bi-sequential (in fact, $Y \times Z$ is not even a k-space).

Proof. Let Y and Z be as in Example 6.5. We shall show that $Y \times Z$ is not a k-space, hence not a quasi-k space, for, $Y \times Z$ is countable, so the countably compact subsets coincide with the compact subsets.

Write $Y = \mathrm{N} \cup \{A\}$, $Z = \mathrm{N} \cup \{B\}$, and define $D = \{(n, n) \in Y \times Z : n \in \mathrm{N}\}$. We shall show that $D \cap K$ is closed in K for every compact subset K of $Y \times Z$, but that D is not closed in $Y \times Z$.

Since $A \cap B \neq \emptyset$, it follows that $\bar{D} = D \cup \{(A, B)\}$, hence D is not closed in $Y \times Z$. Next, to show $D \cap K$ is closed in K for every compact subset K of $Y \times Z$, it suffices to show $D \cap K$ is finite. If $D \cap K$ is infinite, then $\bar{D} \cap K$ must be the one-point compactification of the countable discrete space $D \cap K$, hence there is a sequence $\langle (n_k, n_k) \rangle$ in $D \cap K$ which converges to (A, B). Hence $n_k \to A$ in Y, and $n_k \to B$ in Z. Let $E = \{n_k : k \in \mathrm{N}\}$. Then E' is open in $\beta\mathrm{N} \setminus \mathrm{N}$, by Remark 6.4(e). Since every neighborhood of the set A in $\beta\mathrm{N}$ contains all but finitely many elements of E, it follows that $E' \subset A$; similarly, $E' \subset B$. Hence $\emptyset \neq E' \subset A \cap B$, so the interiors in $\beta\mathrm{N} \setminus \mathrm{N}$ of A and B intersect, contrary to the assumptions of Example 6.5. That completes the proof.

Example 6.6 raises the question: Under what circumstances is the product of two countably bi-sequential spaces again countably bi-sequential? One such condition is provided by the following result.

Proposition 6.7 (E. Michael [21, Proposition 4.D.4]). *If Y is bi-sequential, then $Y \times X$ is countably bi-sequential for every countably bi-sequential space X.*

However, the converse of Proposition 6.7 is false, as Example 6.9 shows. To obtain this example, we need Proposition 6.8 below.

Proposition 6.8. *Let X be a locally compact Hausdorff sequential space, and let Y be countably bi-sequential. If $C \times Y$ is Fréchet (countably bi-*

sequential) for every separable compact $C \subset X$ [9] *then $X \times Y$ is Fréchet (countably bi-sequential). In particular, if every separable compact subset of X is bi-sequential, then $X \times Y$ is countably bi-sequential.*

Proof. By [18, Theorem 4.2], $X \times Y$ is sequential. Let us first show that if $C \times Y$ is Fréchet for every separable compact $C \subset X$, then $X \times Y$ is Fréchet. By [21, Lemma 8.3 and Proposition 8.7], it suffices to show every countable subset of $X \times Y$ is Fréchet; hence it clearly suffices to show $C \times Y$ is Fréchet for every countable subset C of X (since any subspace of a Fréchet space is Fréchet).

Suppose $F \subset C \times Y$, and $(x, y) \in \bar{F} \cap C \times Y$. We must show there is a sequence in F which converges to (x, y). Let K be a compact neighborhood of x in X. Then $(C \cap K)^-$ is a separable compact subset of X, and (x, y) belongs to the closure of $F \cap [(C \cap K)^- \times Y]$ in $(C \cap K)^- \times Y$, by hypothesis a Fréchet space, hence there is a sequence in F which converges to (x, y).

By a result of Michael [21, Proposition 4.D.5], a space Y is countably bi-sequential if and only if $Y \times I$ is Fréchet. If $C \times Y$ is countably bi-sequential for each separable compact $C \subset X$, then $C \times Y \times I$ is Fréchet, hence $X \times Y \times I$ is Fréchet by the previous paragraph, and thus $X \times Y$ is countably bi-sequential.

The "in particular" follows from Proposition 6.7.

Example 6.9 (measurable cardinal). Assuming the existence of a measurable cardinal, there exists a compact Hausdorff space X which satisfies the following conditions.

(i) $X \times Y$ is countably bi-sequential for every countably bi-sequential space Y.

(ii) X is not bi-sequential.

Proof. Assuming the existence of a measurable cardinal, E. Michael [21, Example 10.15] has an example of a one-point compactification X of a discrete space such that X is countably bi-sequential but not bi-sequential. If Y is a countably bi-sequential space, then so is $X \times Y$, since the separable compact subsets of X are finite sets or convergent sequences (which are metrizable and hence bi-sequential).

Remark 6.10. It is not known whether the product of two compact

[9] Actually, with some effort, it can be shown that it suffices to check this condition for compact subspaces C having a dense subset homeomorphic to $\mathbb{N}$.

Hausdorff Fréchet spaces is again Fréchet; see [21, Problem 3]. [10] However, in view of Proposition 6.8 and Footnote 9, if there are two such spaces whose product is not Fréchet, then there are two such spaces that are compactifications of N.

Analogous to Proposition 6.7, E. Michael [21, Proposition 4.E.3] has shown that the product of a countably bi-k-space and a bi-sequential space is a countably bi-k-space. Example 6.6 implies that the product of two countably bi-k-spaces need not be countably bi-k. We now prove the following stronger result, which answers a question raised by Arhangel'skiĭ in a letter to Michael.

Example 6.11 ($2^{\aleph_0} < 2^{\aleph_1}$). There exists a countable, regular, countably bi-sequential space Y and a compact Hausdorff space K such that $Y \times K$ is not singly bi-k. ($Y \times K$ must be a k-space, by a well-known result of D.E. Cohen [6, 3.2].)

Proof. Let $K = \beta N$, let $X = N \cup A$, where A is the subset of $\beta N \setminus N$ in Example 6.5, and let $Y = X/A$ be the quotient space as in Example 6.5. Observe that $X \to Y$ is a perfect map, and that X is completely regular. By [20, Corollary 1.3], it follows that X is homeomorphic to a closed subspace of $Y \times \beta N$, the product of a countably bi-sequential space and a compact Hausdorff space. Since a closed subspace of a singly bi-k-space is again singly bi-k [21, Section 5.E], if $Y \times \beta N$ is singly bi-k, then so is X. We proceed to show X is not singly bi-k.

As in Example 6.5, let $x \in A \cap B$. If X is singly bi-k, then, since $x \in \overline{N}$, there exists a k-sequence $\langle A_n \rangle$ with $x \in (N \cap A_n)^-$ for all n. Suppose $\langle A_n \rangle$ converges to C, some compact subset of X.

Let $B_n = A_n \cap N$. Then $\bigcap_{n=1}^{\infty} B_n^{\beta}$ is a closed G_δ and hence a zero set in βN which contains x. By 6.5 (ii), there exists a point y belonging to $\bigcap_{n=1}^{\infty} B_n^{\beta}$ and the interior in $\beta N \setminus N$ of B. Since the latter does not intersect X, there exists an open-closed subset U of βN which contains y and does not intersect C. Because $y \in \bigcap_{n=1}^{\infty} B_n^{\beta}$, $U \cap B_n \neq \emptyset$ for all n.

[10] Recently, T.K. Boehme and M. Rosenfeld [4], assuming the continuum hypothesis, have obtained an example of two compact Hausdorff Fréchet spaces whose product is not Fréchet. V.I. Malyhin also has such an example, assuming Martin's Axiom, which will be included in a forthcoming paper with B. Šapirovskiĭ. Actually, it can be shown that Boehme and Rosenfeld's example can also be obtained assuming Martin's Axiom instead of the (stronger) continuum hypothesis.

But $\langle A_n \rangle$ converges to C, and $X \setminus U$ is a neighborhood of C, so some $A_n \subset X \setminus U$. Hence

$$\emptyset = A_n \cap U \supset B_n \cap U \neq \emptyset.$$

This contradiction completes the proof that X is not singly bi-k.

7. Miscellaneous examples

E. Michael observed in [21, Problem 6] that in a Hausdorff singly bi-k-space, every G_δ subset is a k-space, and asked whether the converse is true. Assuming $2^{\aleph_0} < 2^{\aleph_1}$, we proceed to show it is not.

Example 7.1 ($2^{\aleph_0} < 2^{\aleph_1}$). There exists a regular Lindelöf space X which is not singly bi-k, such that every G_δ subset of X is a k-space.

Proof. Let $X = \mathrm{N} \cup A$ be the space in the proof of Example 6.11. Then X is a regular Lindelöf space which is not singly bi-k. We shall now show that, because A satisfies Example 6.5 (i) (every zero set intersecting A intersects A^0, the interior of A in $\beta\mathrm{N} \setminus \mathrm{N}$), every G_δ subset of X is a k-space.

Let G be a G_δ subset of X, and suppose $S \subset G$ has the property that for each compact $K \subset G$, $S \cap K$ is compact. It remains to show that S is closed in G.

Suppose otherwise. Let x belong to the closure of S in G, but $x \notin S$. The strategy is as follows. First we produce a subset D of S such that D is closed in S, $D \subset \mathrm{N}$, and x belongs to the closure of D in G; then we produce a compact subset of G whose intersection with D is infinite, reaching a contradiction.

First, $G \cap A$ is a G_δ in the compact Hausdorff space A, hence $G \cap A$ is a k-space (see [3, Chapter II, Corollary to Theorem 3.13] or [21, Proposition 2.E.3]), so $S \cap A$ is closed in $G \cap A$, and consequently in G. Thus $x \notin (S \cap A)^\beta$. By Remark 6.4 (d), there exists an open-closed neighborhood U of x in $\beta\mathrm{N}$ such that $U \cap S \cap A = \emptyset$. Let $D = S \cap U$. Then D is closed in S, $D \cap A = \emptyset$, so $D \subset \mathrm{N}$, and because $U \cap G$ is a neighborhood of x in G, x belongs to the closure of D in G.

Let $G = G^* \cap X$, where G^* is a G_δ subset of $\beta\mathrm{N}$. Since $x \in D^\beta \cap G^*$, a non-empty G_δ in $\beta\mathrm{N}$, it is easy to check (using only the regularity of $\beta\mathrm{N}$) that there exists Z, a closed G_δ, which must be a zero set of $\beta\mathrm{N}$, with $x \in Z \subset D^\beta \cap G^*$. Because $x \in Z \cap A$ (x belongs to the closure of

D, points of $\mathbf{N}$ are open, and $x \notin S$, hence $x \notin D$), and $Z \setminus \mathbf{N}$ is a zero set of $\beta\mathbf{N} \setminus \mathbf{N}$, $(Z \setminus \mathbf{N}) \cap A \neq \emptyset$, so $Z \cap A^0 \neq \emptyset$. But then $Z \cap A^0$ is a non-empty G_δ in $\beta\mathbf{N} \setminus \mathbf{N}$, hence $(Z \cap A^0)^0 \neq \emptyset$, by Remark 6.4 (f). Thus $(Z \cap A)^0$ is not empty, so by Remark 6.4(e), there exists an infinite subset E of $\mathbf{N}$ such that $E' \subset Z \cap A$. Hence

$$E' \subset Z \cap X \subset D^\beta \cap G^* \cap X \subset D^\beta \cap G .$$

Thus $E^\beta \setminus \mathbf{N} \subset (D^\beta \setminus \mathbf{N}) \cap G$; that is, $E' \subset D' \cap G$, so $E \setminus D$ is finite by Remark 6.4(c). Let $F = E \cap D$. Then F is infinite,

$$F^\beta = F' \cup F \subset E' \cup D \subset G , \quad F^\beta \cap D = F .$$

Thus F^β is a compact subset of G whose intersection with D is infinite and hence (being a subset of $\mathbf{N}$) not compact. This implies $F^\beta \cap U$ is a compact subset of G whose intersection with S, namely F, is not compact, a contradiction. That completes the proof.

Remark 7.2. Example 7.1 satisfies Lemma 4.6(ii), but not Lemma 4.6(i).

We now come to our next example. It is known (see [3] or [21, Proposition 2.E.3]) that spaces of pointwise countable type are preserved by perfect pre-images. Consequently, if a space X admits a perfect map onto a first-countable space, then X must be of pointwise countable type. However, the converse is false, as the following example demonstrates.

Example 7.3. There exists a (sequentially compact, locally compact, collectionwise normal) space X of pointwise countable type which does not admit a perfect map onto any space in which every point is a G_δ.

Proof. Let ω_2 be the first ordinal of cardinality $\aleph_2$, and let $X = [0, \omega_2)$ with the order topology. It is easily checked that X is a sequentially compact, locally compact Hausdorff space (hence of pointwise countable type). Suppose there exists a perfect map $f: X \to Y$, with every point of Y a G_δ. We shall show this is impossible.

Since f is perfect and X is countably compact but not compact, Y must be countably compact but not compact, hence Y is uncountable. Let S be a subset of Y with cardinality $\aleph_1$. For $s \in S$, $f^{-1}s$ is compact, hence has cardinality not exceeding $\aleph_1$, so $f^{-1}S$ has an upper bound γ in X. Now $S \subset f[0, \gamma]$, so

$$\aleph_1 = \operatorname{card} S \leqslant \operatorname{card} f[0,\gamma] \leqslant \operatorname{card}[0,\gamma] \leqslant \aleph_1 ,$$

and hence $\operatorname{card} f[0,\gamma] = \aleph_1$. Let $\alpha \in X$ be the first element of X so that $f[0,\alpha]$ has cardinality $\aleph_1$. Note that there is no sequence $\langle \alpha_n \rangle$ increasing to α, for then $f[0,\alpha] = \{f\alpha\} \cup \bigcup_{n=1}^{\infty} f[0,\alpha_n]$ would have cardinality $\aleph_0$.

Since $\{f\alpha\}$ is a G_δ in Y, $f^{-1}f\alpha$ is a G_δ in X, and it follows that there exists $\beta < \alpha$ such that $[\beta,\alpha] \subset f^{-1}f\alpha$ and hence $f[\beta,\alpha] \subset \{f\alpha\}$. Hence $f[0,\beta]$ has cardinality $\aleph_1$, contradicting the assumption that α is the first such. That completes the proof.

Remark 7.4. Dennis Burke has pointed out in a letter that his example [5, Example 3.4] also is a space of pointwise countable type which cannot be mapped perfectly onto a first-countable space. Burke's example is subparacompact (also bi-sequential) but not normal, while Example 7.2 is collectionwise normal but not subparacompact. [11] Neither example is paracompact, and that raises the following question.

Problem 7.5. Is there a paracompact Hausdorff space Y which is of pointwise countable type and does not admit a perfect map onto a first-countable space?

We now come to our final example which shows that, despite Proposition 2.8, $F \leftrightarrow G$ is false in Hausdorff spaces in each of rows 2–5 of Table 1.

Example 7.6. There exists a (sequential) Hausdorff q-space Y which is not singly bi-quasi-k.

Proof. Let Ψ be the space of Isbell [12, 5I, p. 79], where $\Psi = \mathbf{N} \cup D$, $D = \{\omega_E\}_{E\in\mathcal{E}}$ (a set of distinct points not belonging to N), where $\mathcal{E}$ is an uncountable family of infinite subsets of N, maximal with respect to the property that if E_1 and E_2 are distinct members of $\mathcal{E}$, then $E_1 \cap E_2$ is finite. The space Ψ is the weakest T_1-space in which points of N are open and $\{\omega_E\} \cup E$ is open for all $E \in \mathcal{E}$; hence Ψ is a first-countable, locally compact Hausdorff space in which N is dense, and D is a closed, discrete subspace of Ψ.

Let us also observe that two disjoint countable subsets of D can be

[11] Recall that a Hausdorff space is paracompact if and only if it is collectionwise normal and subparacompact [5, Theorem 1.2].

separated by open subsets of Ψ which contain no points of D other than those in the two countable subsets. Indeed, if $A = \{\omega_{E_1}, \omega_{E_2}, \ldots\}$ is a countable subset of D, then

$$\{\{\omega_{E_n}\} \cup E_n \setminus (E_0 \cup E_1 \cup \ldots \cup E_{n-1})\}_{n=1}^{\infty} \quad (E_0 = \emptyset)$$

is a disjoint open (in Ψ) cover of A which separates the points of A.

Let Q denote the rationals with the usual topology, let A be any countable subset of D, and let $g: A \to \mathbf{Q}$ be a (necessarily continuous) bijection. Define $Y = \Psi \cup_g \mathbf{Q}$ to be the adjunction space, given as the quotient of the disjoint union $\Psi + \mathbf{Q}$ under the identification $q = g^{-1}q$ for $q \in \mathbf{Q}$. As a closed subspace of Y, $D = \mathbf{Q}$ + (uncountable discrete space). It follows from the preceding paragraph that Y is a Hausdorff space. Since Y is a quotient of the first-countable space $\Psi + \mathbf{Q}$, Y is sequential.

Next, we show that Y is a q-space. Points of N and of $D \setminus A$ have compact neighborhoods in Y, so Y is certainly a q-space at these points. It remains to consider points belonging to A. Now, D is first-countable as a subspace of Y and each point of Y has a countable neighborhood, hence is a G_δ; so if $a \in A$, let $\langle U_n \rangle$ be a decreasing sequence of neighborhoods of a in Y such that $\langle U_n \cap D \rangle$ is a base for the neighborhoods of a in D and $\bigcap_{n=1}^{\infty} U_n = \{a\}$. If $y_n \in U_n$ for all n, then there are two possibilities: Either infinitely many $y_n \in \mathbf{N}$, in which case infinitely many y_n belong to some $E \in \mathcal{E}$ and $\langle y_n \rangle$ accumulates at ω_E in Ψ, hence in Y; or, if all but finitely many $y_n \in D$, then $y_n \to a$. Therefore Y is a q-space.

We now show that Y is not a Fréchet space. Let $E \in \mathcal{E}$ satisfy $\omega_E \in A$. Then the closure of $\mathbf{N} \setminus E$ in Ψ is $(D \setminus \{\omega_E\}) \cup \mathbf{N} \setminus E$. Since $\mathbf{Q} \setminus \{g(\omega_E)\}$ is dense in Q, the closure of $\mathbf{N} \setminus E$ in Y is $Y \setminus E$. If Y were a Fréchet space, then there would exist a sequence in $\mathbf{N} \setminus E$ converging to ω_E in Y. Since Y is a Hausdorff space, such a seqeunce can have no accumulation points other than ω_E. But (by the maximality condition on $\mathcal{E}$) such a sequence must have a subsequence in some $E' \in \mathcal{E} \setminus \{E\}$, in which case $\omega_{E'}$ is an accumulation point other than ω_E. Hence Y is not a Fréchet space.

Since each point of Y is a G_δ in Y and Y is not Fréchet, Proposition 2.10 implies that Y is not singly bi-k. To conclude that Y is not even singly bi-quasi-k, it will suffice to show that every countably compact subset of Y is countable and therefore compact. If C is a countably compact subset of Y, then the closed subset $C \setminus (\mathbf{N} \cup A)$ of C is countably compact and discrete, hence finite. Since $\mathbf{N} \cup A$ is countable, C must be countable. That completes the proof.

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